Analytics of the Eaton and Kortum model

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1 Assumptions

- I countries, $i = 1...I$
- A continuum of goods $j \in [0, 1]$
- Total consumption is a CES aggregate over goods:

$$U_n = \left[\int_0^1 Q_n(j)^{\frac{\sigma-1}{\sigma}} dj\right]^{\frac{\sigma}{\sigma-1}}$$

- Good j produced with a bundle of inputs, assumed homogenous across commodities, which cost is called c_i (and taken as exogenous initially)
- Production function has constant returns to scale
- Country i's efficiency in producing good j, $z_i(j)$, is the realization of a random variable Z_i drawn (independently for each commodity) from a country-specific distribution, assumed Fréchet (Type II extreme value):

$$F_i(z) = Pr[Z_i \leq z] = e^{-T_i z^{-\theta}}$$

- Perfect competition among producers of good j
- Iceberg trade costs d_{ni} with $d_{ii} = 1$ and $d_{ni} \leq d_{nk} d_{ki}$

2 Results

- Cost of delivering a unit of good j produced in country i to country n:

$$p_{ni}(j) = \frac{c_i}{z_i(j)} d_{ni}$$

It is a realization of the random variable $P_{ni} = \frac{c_i}{Z_i} d_{ni}$ which distribution is given by:

$$G_{ni}(p) = Pr[P_{ni} \leq p] = Pr\left[Z_i \geq \frac{c_i d_{ni}}{p}\right] = 1 - e^{-T_i\left(\frac{c_i d_{ni}}{p}\right)^{-\theta}}$$

- Perfect competition implies that the price consumers in country n actually pay for good j is the lowest across all sources i:

$$p_n(j) = min\{p_{ni}(j); i = 1...I\}$$

It is the realization of the random variable $P_n = min\{P_{ni}; i = 1...I\}$ which distribution is given by:

$$\begin{aligned} G_n(p) &= Pr[P_n \leq p] = 1 - \prod_{i=1}^{I} Pr[P_{ni} > p] \\ &= 1 - \prod_{i=1}^{I} [1 - G_{ni}(p)] = 1 - \prod_{i=1}^{I} e^{-T_i\left(\frac{c_i d_{ni}}{p}\right)^{-\theta}} \\ &= 1 - e^{-p^\theta \sum_{i=1}^{I} T_i (c_i d_{ni})^{-\theta}} \equiv 1 - e^{-p^\theta \Phi_n} \end{aligned}$$

- Probability that country i provides a good at the lowest price in country n:

  - If $p_{ni}(j) = p$ then the probability that i is the lowest cost supplier is:

$$\begin{aligned} \prod_{s \neq i} Pr[P_{ns} \geq p] &= \prod_{s \neq i} [1 - G_{ns}(p)] \\ &= \prod_{s \neq i} e^{-T_s (c_s d_{ns})^{-\theta} p^\theta} \\ &= e^{-p^\theta \sum_{s \neq i} T_s (c_s d_{ns})^{-\theta}} \end{aligned}$$

- Integrate over all prices:

$$
\begin{aligned}
\pi_{ni} &= \int_0^\infty e^{-p^\theta \sum_{s\neq i} T_s(c_s d_{ns})^{-\theta}} dG_{ni}(p) \\
&= \int_0^\infty e^{-p^\theta \sum_{s\neq i} T_s(c_s d_{ns})^{-\theta}} T_i(c_i d_{ni})^{-\theta}\theta p^{\theta-1} e^{-T_i(c_i d_{ni})^{-\theta}p^\theta} d(p) \\
&= \frac{T_i(c_i d_{ni})^{-\theta}}{\Phi_n} \int_0^\infty \Phi_n e^{-p^\theta \Phi_n} \theta p^{\theta-1} dp \\
&= \frac{T_i(c_i d_{ni})^{-\theta}}{\Phi_n} \left[1 - G_n(p)\right]_0^\infty \\
&= \frac{T_i(c_i d_{ni})^{-\theta}}{\Phi_n}
\end{aligned}
$$

- Price of a good that n actually buys from i has the following distribution:

  - n buys from i if and only if it is the lowest cost supplier

  - if this price is q then the probability that this happens is:

$$
Pr[q \leq P_{ns}(j); s \neq i] = \prod_{s\neq i} Pr[P_{ns}(j) \geq q] = e^{-q^\theta \sum_{s\neq i} T_s(c_s d_{ns})^{-\theta}}
$$

  - Integrating over all prices:

$$
\begin{aligned}
\int_0^p e^{-q^\theta \sum_{s\neq i} T_s(c_s d_{ns})^{-\theta}} dG_{ni}(p) &= \int_0^p e^{-q^\theta \sum_{s\neq i} T_s(c_s d_{ns})^{-\theta}} T_i(c_i d_{ni})^{-\theta}\theta q^{\theta-1} e^{-q^\theta T_i(c_i d_{ni})^{-\theta}} dq \\
&= \int_0^p e^{-q^\theta \Phi_n} T_i(c_i d_{ni})^{-\theta}\theta q^{\theta-1} dq \\
&= \frac{T_i(c_i d_{ni})^{-\theta}}{\Phi_n} \int_0^p \Phi_n e^{-q^\theta \Phi_n} \theta q^{\theta-1} dq \\
&= \frac{T_i(c_i d_{ni})^{-\theta}}{\Phi_n} \left[1 - e^{-q^\theta \Phi_n}\right]_0^p \\
&= \pi_{ni} G_n(p)
\end{aligned}
$$

$\Rightarrow$ Distribution of the price charged by i in n conditionally on selling in n:

$$
\frac{1}{\pi_{ni}} \int_0^p e^{-q^\theta \sum_{s\neq i} T_s(c_s d_{ns})^{-\theta}} dG_{ni}(q) = G_n(p)
$$

- CES price index:

$$P_n^{1-\sigma} = \int_0^1 P_n(j)^{1-\sigma} dj$$

  – Ex ante:

$$\begin{aligned} P_n^{1-\sigma} &= \int_0^\infty p^{1-\sigma} dG_n(p) \\ &= \int_0^\infty p^{1-\sigma} \theta p^{\theta-1} \Phi_n e^{-p^\theta \Phi_n} dp \end{aligned}$$

  – define $x = \Phi_n p^\theta \Rightarrow dx = \Phi_n \theta p^{\theta-1} dp$

$$\begin{aligned} P_n^{1-\sigma} &= \int_0^\infty (x/\Phi_n)^{\frac{1-\sigma}{\theta}} e^{-x} dx \\ &= \Phi_n^{\frac{\sigma-1}{\theta}} \int_0^\infty x^{\frac{1-\sigma}{\theta}} e^{-x} dx \\ &= \Gamma\left(1 + \frac{1-\sigma}{\theta}\right) \Phi_n^{\frac{\sigma-1}{\theta}} \end{aligned}$$

  – Thus the price index:

$$P_n = \left[\Gamma\left(\frac{\theta+1-\sigma}{\theta}\right)\right]^{\frac{1}{1-\sigma}} \Phi_n^{-1/\theta} \equiv \gamma \Phi_n^{-1/\theta}$$

3 The gravity equation

- Share of country n's expenditures imported from i:

$$\frac{X_{ni}}{X_n} = \pi_{ni} = \frac{T_i (c_i d_{ni})^{-\theta}}{\Phi_n}$$

- i's total sales:

$$Y_i = \sum_n X_{ni} = T_i c_i^{-\theta} \sum_{m=1}^{M} \frac{d_{mi}^{-\theta} X_m}{\Phi_m} \equiv T_i c_i^{-\theta} \Omega_i^{-\theta}$$

$\Rightarrow$ Gravity equation:

$$X_{ni} = \gamma^{-\theta} X_n P_n^{\theta} Y_i \Omega_i^{\theta} d_{ni}^{-\theta}$$

- country i's normalized import share:

$$S_{ni} = \frac{X_{ni}/X_n}{X_{ii}/X_i} = \underbrace{d_{ni}^{-\theta}}_{Distance} \underbrace{\frac{\Phi_i}{\Phi_n}}_{Comp.\ Advantages}$$

- Without trade barriers, $S_{ni} = 1$