Lecture 3: The 2x2x2 Heckscher-Ohlin-Samuelson Model

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Outline of Lecture 3

- Autarky
- Free Trade Equilibrium: Small Open Economy
- Free Trade Equilibrium: Two-Country World
- The Lerner Diagram
- Duality Tools

Overview

- Different factor endowments, *same technologies*
- Countries have comparative advantage in sectors using their abundant factors intensively
- They gain from trade by reallocating output to comparative advantage sectors
- The HOS model provides answers to 4 questions:

(H-O) what is the pattern of trade?
(FPE) how does trade affect factor prices?
(S-S) if prices change, how do factor prices change?
(R) if endowments change, how do outputs change?

Assumptions

- 2 goods (1 and 2), 2 factors (K and L), 2 countries (H and F)
- *Same* technology in both countries: $y_i = f_i(K_i, L_i)$ with $f(\cdot)$ increasing, concave, and linearly homogenous (CRS).
- Factors are fully mobile across sectors and fully immobile across countries.
- Perfectly competitive goods and factor markets.
- Identical convex and homothetic preferences in both countries.

- Firms maximize profits at market prices.

$$\max_{K_i,L_i}\{p_i f_i(K_i,L_i) - rK_i - wL_i\} \Rightarrow p_i \frac{\partial f_i}{\partial K_i} = r$$
$$p_i \frac{\partial f_i}{\partial L_i} = w$$

- At the optimum, the Marginal Rate of Technological Substitution is equal to the relative factor price:

$$MRTS_i \equiv \frac{\frac{\partial f_i}{\partial K_i}}{\frac{\partial f_i}{\partial L_i}} = \frac{w}{r}, i = 1,2$$

- Due to the CRS assumption, the optimal K/L ratio does not depend on the scale.

The MRTS is also known as the Technical Rate of Substitution (TRS).

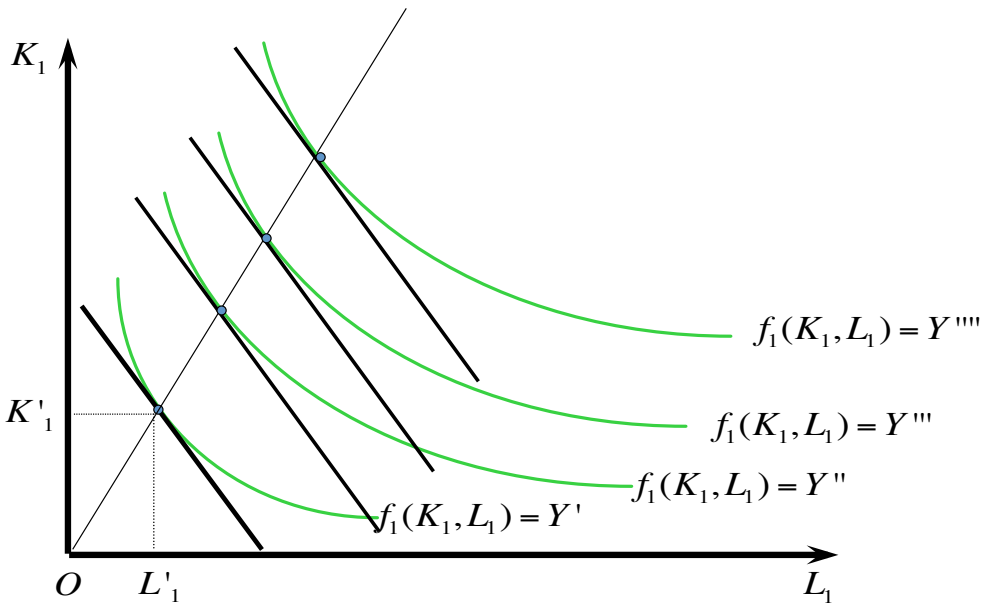
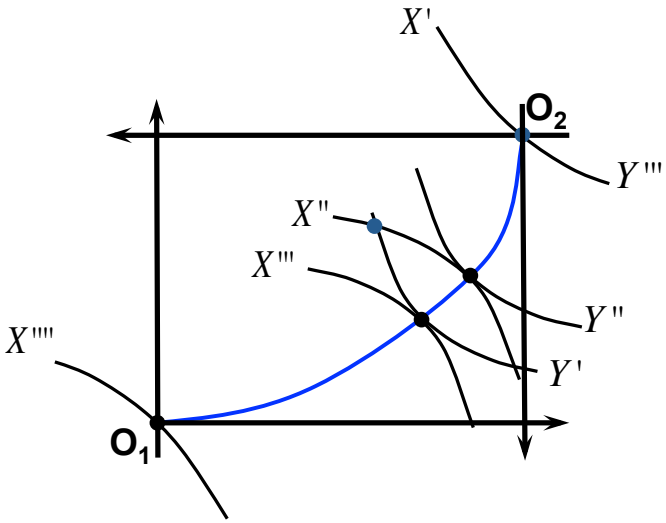
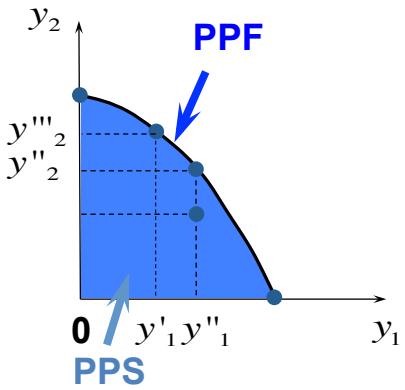


Figure: Optimal input choice minimizes costs subject to a production constraint

- Cost minimization in both sectors defines efficient production plans.
- These efficient plans yield a Production Possibility Frontier.
- The PPF is concave. The Production Possibility Set is convex.





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The Marginal Rate of Transformation

- The Marginal Rate of Transformation equals the slope of the PPF.
- It captures the opportunity cost of producing an extra unit of good 1 in units of good 2.
- At full employment, along the PPF we have $dL_1 = -dL_2$ so that

$$\frac{dy_1}{dy_2} = -\frac{\frac{\partial f_1}{\partial K_1}}{\frac{\partial f_2}{\partial K_2}}$$

and similarly

$$\frac{dy_1}{dy_2} = -\frac{\frac{\partial f_1}{\partial L_1}}{\frac{\partial f_2}{\partial L_2}}$$

- Profit maximization implies

$$MRT_{12} \equiv \frac{dy_1}{dy_2}|_{y_1, y_2 \in PPF} = -\frac{p_2}{p_1}$$

Autarky Equilibrium Conditions

- Utility maximization (MRS = relative price)

$$\frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = \frac{p_2}{p_1}$$

- Goods markets clearing

$$x_i = f_i(K_i, L_i), i = 1, 2$$

- Cost minimization (MRTS=relative factor price)

$$\frac{\frac{\partial f_i}{\partial K_i}}{\frac{\partial f_i}{\partial L_i}} = \frac{w}{r}, i = 1, 2$$

- Factor markets clearing (full employment)

$$L_1 + L_2 = L$$
$$K_1 + K_2 = K$$

- Marginal cost pricing (zero-profit condition)

$$p_i y_i = wL_i + rK_i, i = 1, 2$$

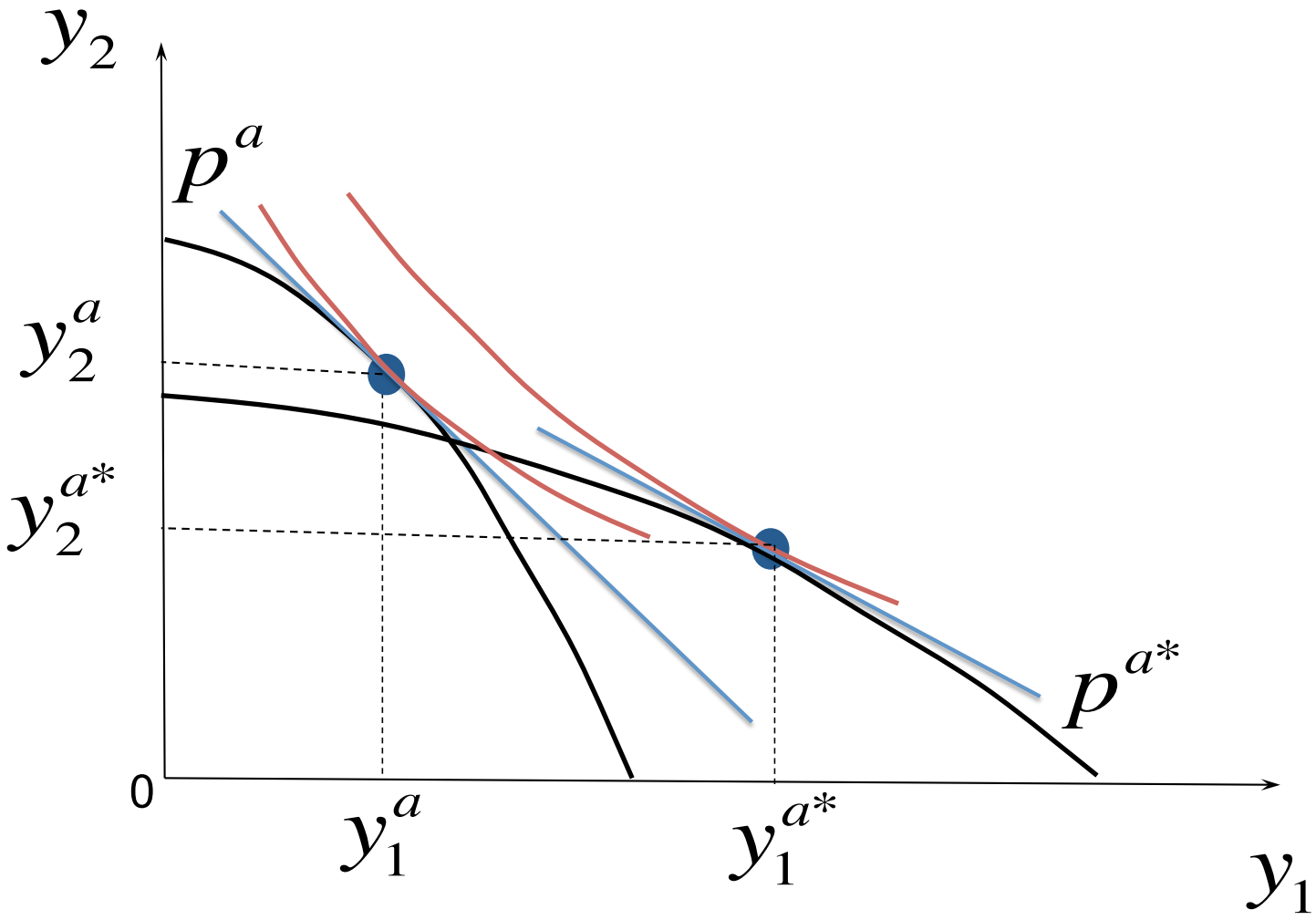


Figure: Autarky equilibria in two countries that have identical technologies and preferences, but different factor endowments.

Free Trade Equilibrium

- We consider the polar case of free and costless trade: prices are equal everywhere.
- We take a two-step approach to derive the free trade equilibrium:
  - small-open economy (SOE), where exogenous world prices apply
  - two-country world: endogenous prices, *world* goods markets clear

Free Trade Equilibrium: Small Open Economy

- Prices p_1 and p_2 are exogenously given.
- Cost minimization (MRTS=relative factor price)

$$\frac{\frac{\partial f_i}{\partial K_i}}{\frac{\partial f_i}{\partial L_i}} = \frac{w}{r}, i = 1, 2$$

- Factor markets clearing (full employment)

$$L_1 + L_2 = L$$
$$K_1 + K_2 = K$$

- Marginal cost pricing (zero profit)

$$p_i = wL_i + rK_i, i = 1, 2$$

Free Trade Equilibrium: Small Open Economy

- Define $a_{vi}(w,r), i = 1,2, v = K, L$ and $c_i(w,r)$ such that

$$c_i(w,r) = \min_{L_i,K_i}\{wL_i + rK_i : f_i(K_i,L_i) \geq 1\} \equiv wa_{Li}(w,r) + ra_{Ki}(w,r)$$

- Then a SOE free trade equillbrium satisfies:
  - Factor markets clearing (full employment)

$$\begin{aligned} a_{L1}(w,r)y_1 + a_{L2}(w,r)y_2 &= L \\ a_{L2}(w,r)y_1 + a_{K2}(w,r)y_2 &= K \end{aligned} \tag{FE}$$

  - Marginal cost pricing (zero profit)

$$\begin{aligned} p_1 &= c_1(w,r) \\ p_2 &= c_2(w,r) \end{aligned} \tag{ZP}$$

- (FE)-(ZP) form a system of 4 equations in 4 unknowns: w, r, y_1, y_2.

Free Trade Equilibrium: Small Open Economy

Lemma (Factor Price Insensitivity)

(ZP) has a unique solution $\{w, r\}$ that depends only on prices $\{p_1, p_2\}$, not endowments $\{K, L\}$ if:

- *both sectors produce ('diversification')*
- *technologies do not exhibit Factors Intensity Reversals (FIR's), e.g.* $\frac{a_{L1}(w,r)}{a_{K1}(w,r)} > \frac{a_{L2}(w,r)}{a_{K2}(w,r)}, \forall w, r$

- Plugging the unique $\{w, r\}$ in (FE) yields $\{y_1, y_2\}$.
- Factor prices are 'insensitive' to endowments:
  - this would not hold in a one-sector economy, e.g. extra L supply would require a fall in w to be 'absorbed'
  - in a two-sector economy, the extra L is 'absorbed' at the same w by reallocating output towards the L-intensive sector.

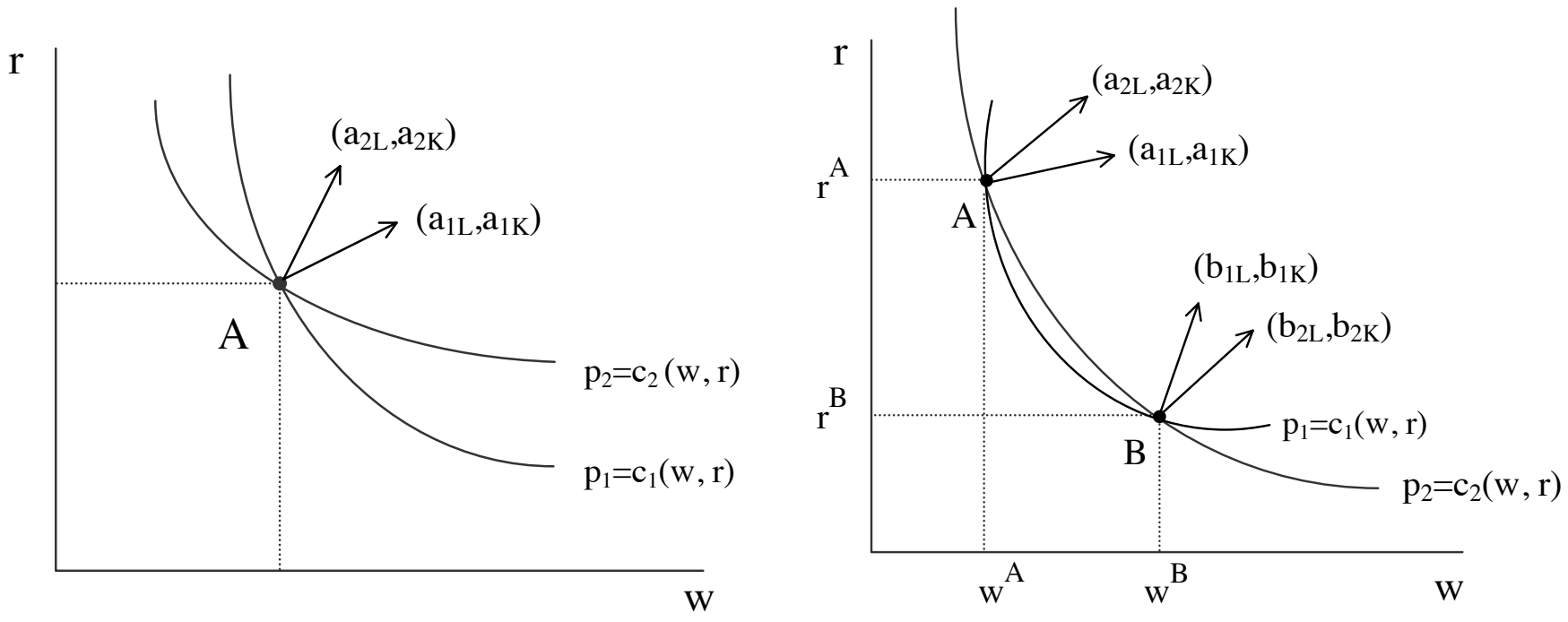


Figure: Equilibrium factor prices without (left) and with (right) Factor Intensity Reversals. The tangent to the isocost curve has slope $\frac{a_{Li}(w,r)}{a_{Ki}(w,r)}$.

Factor Price Equalization

Theorem (Factor Price Equalization)

Under the same prices and technologies, if both goods are produced and FIR's do not occur, then a small open economy has the same factor prices as the rest of the world.

- From the Lemma: if the SOE and the ROW have the same (ZP) and there is a unique solution $\{w, r\}$, then it must be the same.
- Note that factor prices are equalized *without* any cross-border factor movements.

Comparative Statics: Changes in Product Prices

Totally differentiating (ZP) yields:

$$dp_i = a_{Li}dw + a_{Ki}dr \Rightarrow \underbrace{\frac{dp_i}{p_i}}_{\hat{p}_i} = \underbrace{\frac{wa_{Li}}{c_i}}_{\theta_{iL}} \underbrace{\frac{dw}{w}}_{\hat{w}} + \underbrace{\frac{ra_{Ki}}{c_i}}_{\theta_{iK}} \underbrace{\frac{dr}{r}}_{\hat{r}}$$

Denoting by Θ the cost share matrix:

$$\begin{vmatrix} \hat{p}_1 \\ \hat{p}_2 \end{vmatrix} = \begin{vmatrix} \theta_{1L} & \theta_{1K} \\ \theta_{2L} & \theta_{2K} \end{vmatrix} \begin{vmatrix} \hat{w} \\ \hat{r} \end{vmatrix} \Rightarrow \begin{vmatrix} \hat{w} \\ \hat{r} \end{vmatrix} = \frac{1}{|\Theta|} \begin{vmatrix} \theta_{2K} & -\theta_{1K} \\ -\theta_{2L} & \theta_{1L} \end{vmatrix} \begin{vmatrix} \hat{p}_1 \\ \hat{p}_2 \end{vmatrix}$$

Theorem (Stolper-Samuelson, 1941)

A rise in the relative price of a good will increase the real return to the factor used intensively in that good, and reduce the real return of the other factor.

For example if 1 is labor-intensive then $\theta_{1L} - \theta_{2L} > 0$ and:

$$\hat{w} > \hat{p}_1 > \hat{p}_2 > \hat{r}$$

Comparative Statics: Changes in Endowments

Rewriting and totally differentiating (FE):

$$a_{v1}dy_1 + a_{v2}dy_2 = dV_v, v = K, L$$

Denote factor shares by $\lambda_{iv} = \frac{y_i a_{vi}}{V_v}$ and the factor share matrix by Λ:

$$\begin{vmatrix} \hat{L} \\ \hat{K} \end{vmatrix} = \begin{vmatrix} \lambda_{1L} & \lambda_{2L} \\ \lambda_{1K} & \lambda_{2K} \end{vmatrix} \begin{vmatrix} \hat{y_1} \\ \hat{y_2} \end{vmatrix} \Rightarrow \begin{vmatrix} \hat{y_1} \\ \hat{y_2} \end{vmatrix} = \frac{1}{|\Lambda|} \begin{vmatrix} \lambda_{2K} & -\lambda_{2L} \\ -\lambda_{1K} & \lambda_{1L} \end{vmatrix} \begin{vmatrix} \hat{L} \\ \hat{K} \end{vmatrix}$$

Theorem (Rybczynski, 1955)

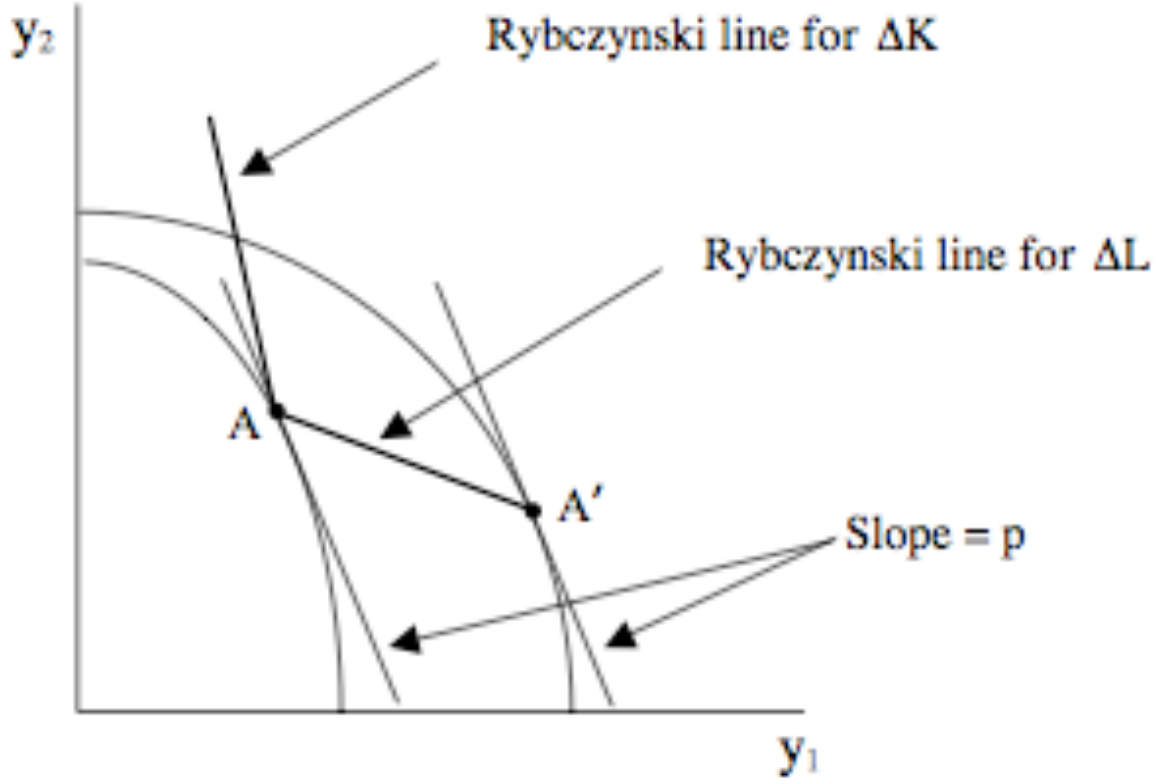
An increase in a factor endowment will increase the output of the industry using it intensively, and reduce the output of the other industry.

For example suppose that 1 is labour-intensive. Then:

$$\hat{L} > 0, \hat{K} = 0 \Rightarrow \hat{y_1} > \hat{L} > 0 > \hat{y_2}$$
$$\hat{K} > 0, \hat{L} = 0 \Rightarrow \hat{y_2} > \hat{K} > 0 > \hat{y_1}$$

Rybczynski Lines

Growth in the endowment of one factor creates the 'Rybczynski line':



The Heckscher-Ohlin Theorem

Theorem (Heckscher-Ohlin Theorem)

Each country will export the good that uses its abundant factor intensively.

Sketch of the proof using the Rybczynski theorem:

- Suppose the SOE is labor-abundant relative to the ROW.
- According to the Rybczinski theorem, the relative output in the L-intensive sector (say, sector 1) must be greater than in the ROW.
- National goods markets clearing and identical preferences imply that the relative consumption of good 1 is greater than in the ROW.
- Therefore under autarky the relative price of good 1 is lower than in the ROW.
- As the Home country becomes a SOE, the relative price of good 1 increases which reallocates output towards sector 1, and consumption towards sector 2.

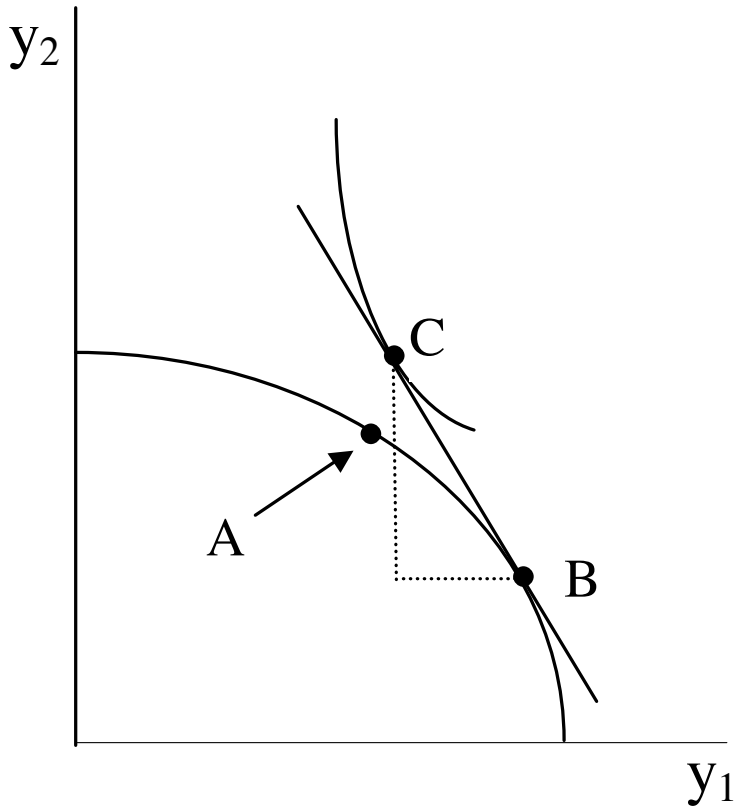


Figure: Labor abundance implies that the Home country will face a higher relative price of good 1 (L-intensive) under free trade, and will start exporting good 1.

Free Trade Equilibrium: Two-Country World

- Consider two countries, Home and Foreign.
- In a two-country world, prices are determined endogenously on world markets.
- The world market clearing condition replaces national market clearing conditions.
- Do the 4 theorems carry over from the SOE case?

The Integrated Economy Approach

- Thought experiment : consider a world economy where both goods *and* factors can move costlessly.
- Then both goods and factor prices must be equal worldwide.
- Denote by ω the vector of factor prices, $A(\omega)$ the matrix of $a_{vi}(w, r)$'s, y the vector of outputs, p the vector of goods prices and $\alpha(p)$ the budget shares.
- An integrated economy equilibrium (IEE) satisfies

$$p = A(\omega)'\omega \tag{ZP}$$
$$y = \alpha(p)\omega' V \tag{GM}$$
$$V = A(\omega)y \tag{FE}$$

- Can free trade in goods replicate an IEE?

The Integrated Economy Approach

- Consider the following definition of a free trade equilibrium

$$p = A(\omega^c)'\omega^c, c = H, F$$
$$y^H + y^F = \alpha(p)(w^H V^H + w^F V^F)$$
$$V^c = A(\omega^c)y^c, c = H, F$$

- Define the FPE set as the set of endowments if v^H, v^F are such that $\exists(y_1^c, y_2^c) \geq 0, v^c = A(\omega)y^c, c = H, F$.
- Then it can be shown that a free trade equilibrium replicates the IEE if endowments are in the FPE set.

- If the v's are in the FPE set, then national FE conditions hold, then the world FE condition holds.
- As factor prices are the same national goods market conditions can be factored as in the world goods market condition.
- The zero profit conditions are the same as in the IEE.
- We can represent this equivalence in a two-country Edgeworth box in endowment space.

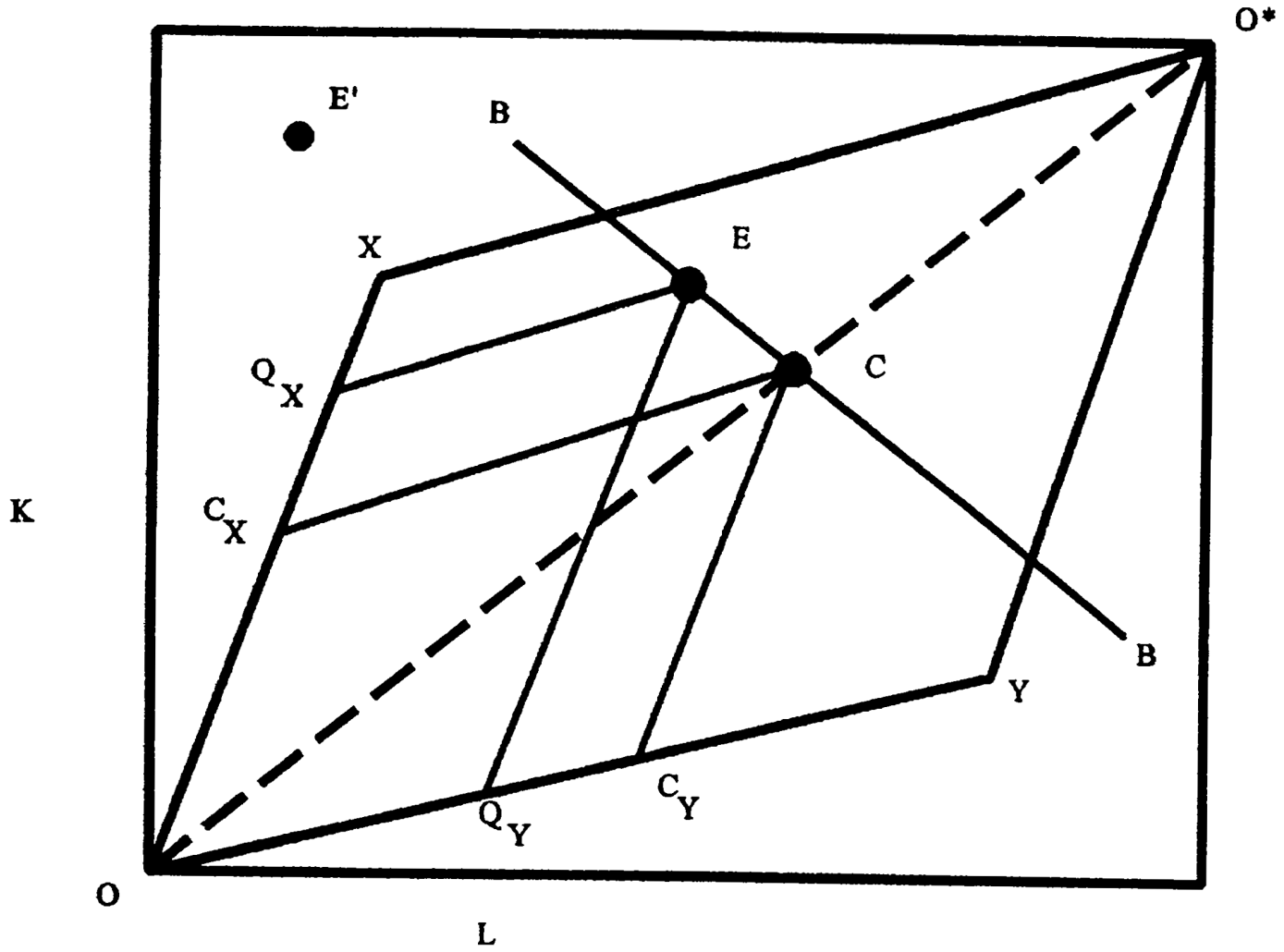


Figure: Factor use at a free trade equilibrium. See next slide.

- At the IEE factor prices are equal and both goods are produced.
- Suppose the integrated economy produces good 1 at X and good 2 at Y. OX and OY represent (cost-minimizing) factor use of each sector.
- Is there some (exogenous) distribution of national endowments such that the IEE can be replicated?
  - at E, the parallelogram $OQ_X Q_Y O^*$ and its mirror image represents output and factor allocation consistent with full employment, common factor prices
  - the same applies to all points in the parallelogram OXO^*Y
  - at point E' country H specializes in the K-intensive good. Outside OXO^*Y doesn't hold.
- Factor use embodied in consumption lies on the diagonal (see Appendix).
- if (BB) has slope $-\frac{w}{r}$ then C represents factors embodied in consumption. EC represents implied net factor trade.
- OXO^*Y is the 'FPE set' and OXY the 'cone of diversification'.

The Integrated Economy and the SOE Approach

- When endowments are in the FPE set:
  - FPE obtains
  - the HO theorem applies: for example if H is K-abundant, then endowments are above the diagonal and H consumes more labor, less capital than it produces.
  - the Rybczinski and Stolper-Samuelson results apply as local results, at equilibrium prices.
- When endowments are not in the FPE set, additional conditions are necessary: FIRs, demand functions...

The Lerner Diagram

- In 1952 Abraham Lerner invented a diagram that summarizes the HOS model and its main predictions.
- The diagram, drawn in factor space, plots:
  - *isovalue* curves (not isoquants), i.e. combinations of factors that yield 1 euro's worth of output, at given prices
  - *isocost* curves, i.e. combinations of factors that cost 1 euro at given factor prices
- The tangency points between both curves represents cost-minimizing factor use in each sector.
- For example, an increase in the relative price of capital tilts factor use towards labor.

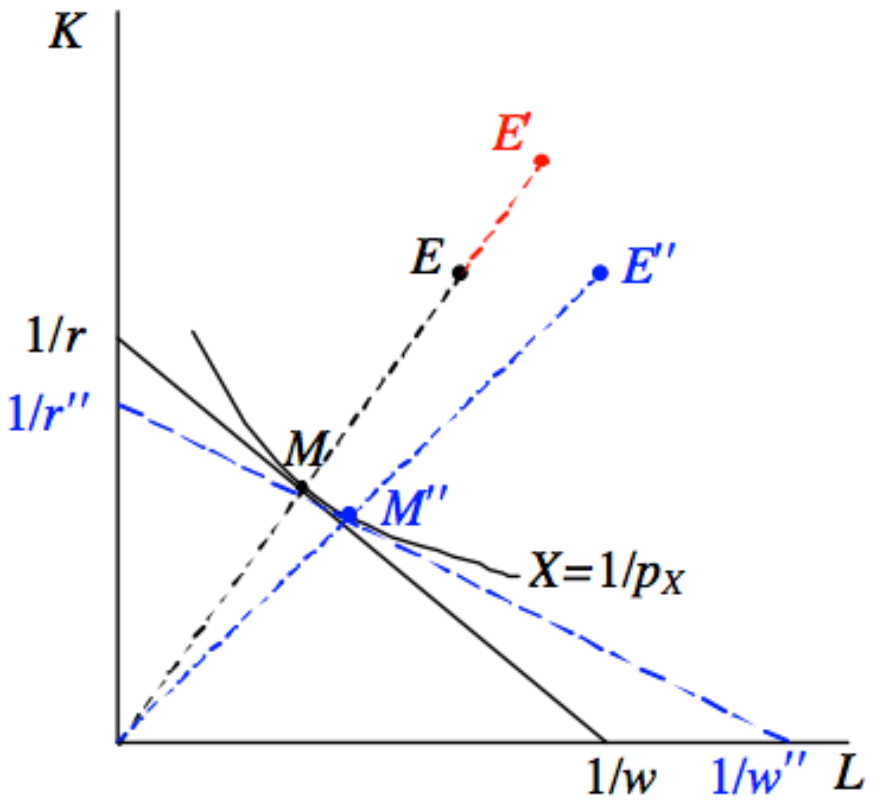
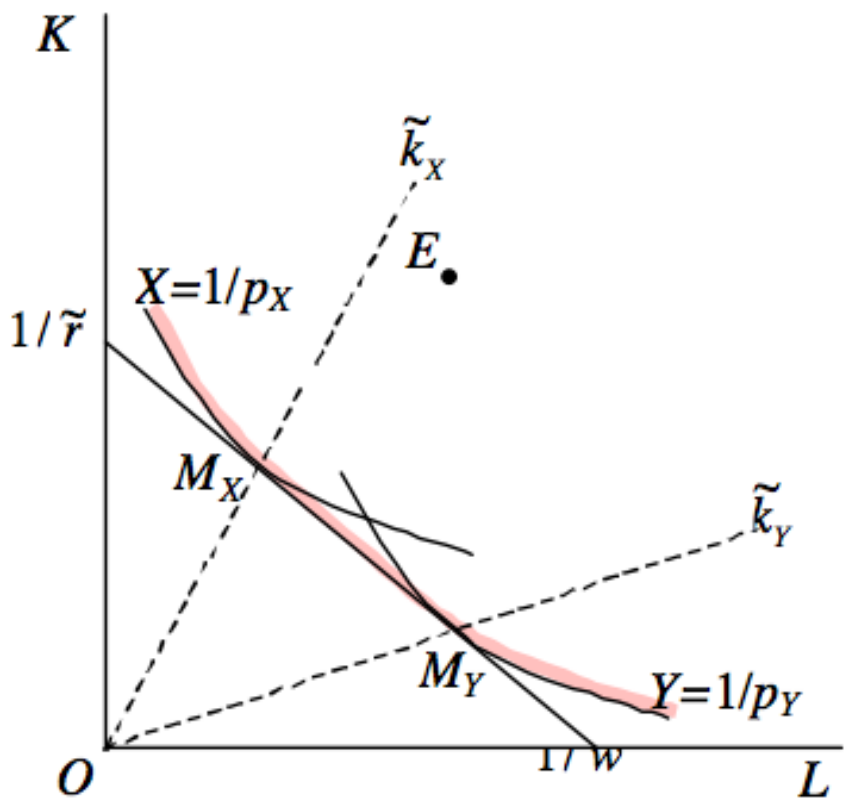
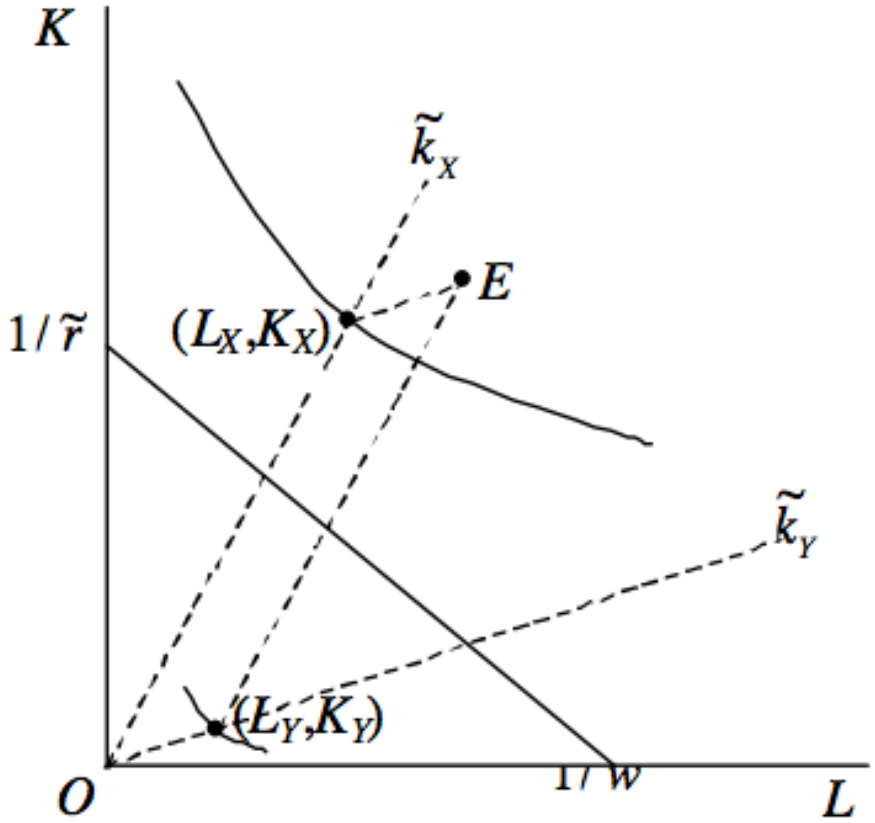


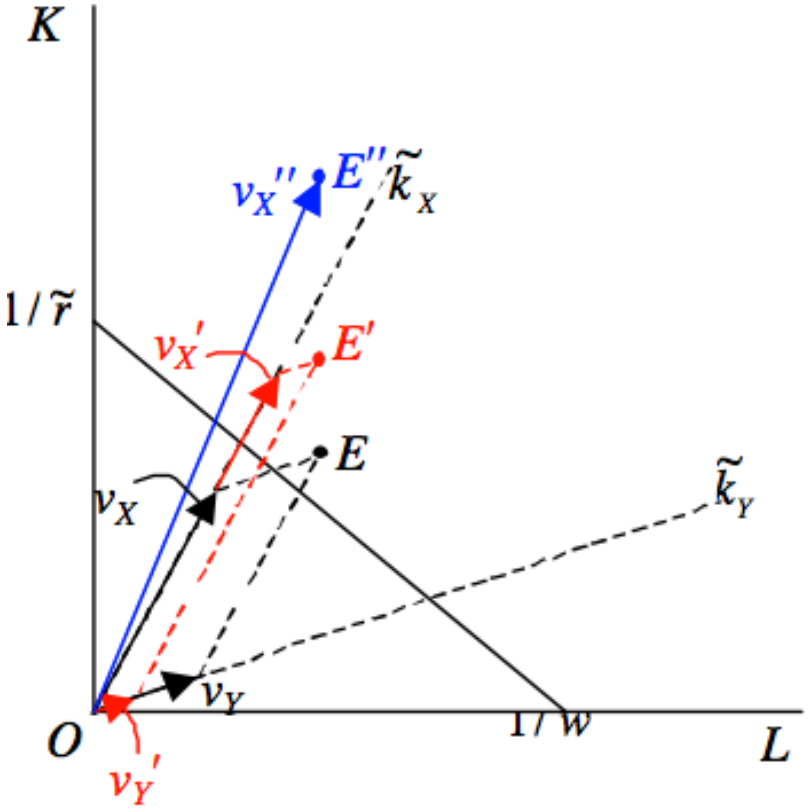
Figure: One-sector Lerner Diagram: effect of a change in factor prices.

- With 2 sectors, the *convex hull* in red represents efficient factor use.
- Efficient factor use is consistent with diversification if endowments are in the diversification cone (as in point E).

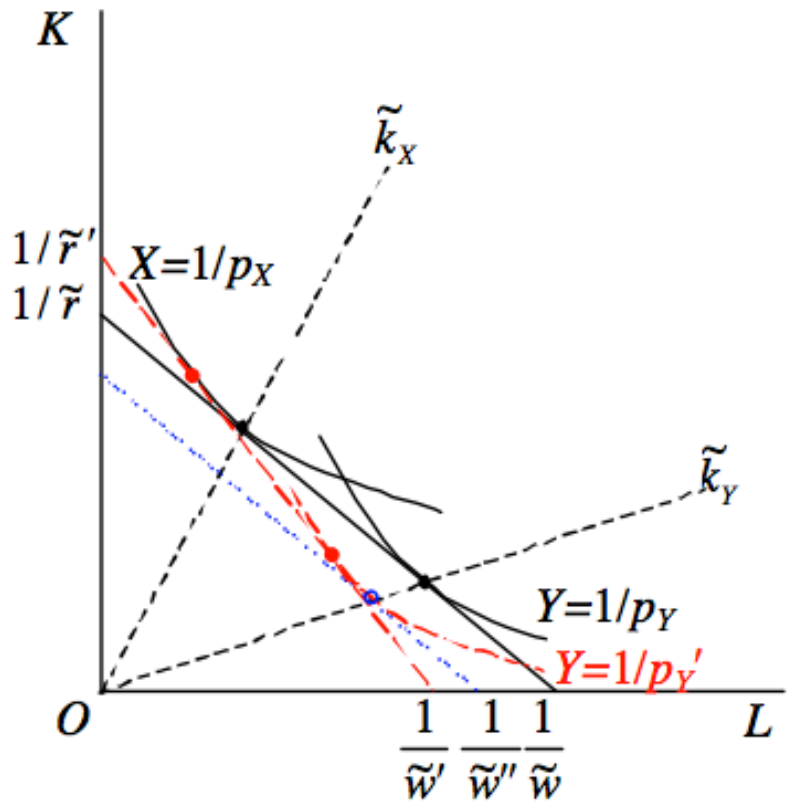




- Suppose that K increases while L and all prices remain constant.
- Output increases in the K-intensive sector (X) and decreases in the other sector (Y).
- The Rybczyinski result still holds when endowments move outside the diversification cone (output of Y is zero).



- Suppose now that the price of the L-intensive good (p_Y) increases, while p_X remains constant.
- The diagram illustrates the Stolper-Samuelson result:
  - in nominal terms w rises and r falls, so that $\frac{w}{r}$ rises
  - in terms of good X the same holds, since p_X remains constant
  - in terms of good Y the real rental falls, since r falls and p_Y rises
  - w rises by more than p_Y, as can be seen from $\tilde{w}'' < \tilde{w}'$



Duality

- Cost minimization and expenditure minimization are dual problems to utility maximization and profit maximization, respectively.
- Several results of neoclassical trade theory can be framed as implications of GE theory, using duality concepts.

Additional References:

- Mas-Colell et al. (1995), Microeconomic Theory, Oxford University Press, chapters 3F, 3G, 5C
- Dixit and Norman (1980), Theory of International Trade, Cambridge University Press, chapter 2.

Duality tools

TABLE I

Schematic comparison of the different functions

		(1) Consumption x	(2) Net Imports m	(3) Factor Content of Net Imports M
(1)	Direct utility function	$u(x)$	$U(m, v) = \text{Max}\,[u(x)\colon F(x-m, v) \leqq 0]$	$\bar{U}(M, v) = \text{Max}\,[u(x)\colon F(x, v+M) \leqq 0]$
(2)	maximised subject to:	$p \cdot x \leqq I$	$p \cdot m \leqq b$	$w \cdot M \leqq b$
(3)	→ Marshallian demand functions	$x(p, I)$	$m(p, b, v)$	$M(w, b, v)$
(4)	→ Indirect utility function	$V(p, I) = u[x(p, I)]$	$H(p, b, v) = U[m(p, b, v), v]$	$L(w, b, v) = \bar{U}[M(w, b, v), v]$
(5)			$= V[p, g(p, v) + b]$	$= H[c(w), b, v]$
(6)	→ Roy's Identity	$V_p(p, I) = -V_I(p, I)x(p, I)$	$H_p(p, b, v) = -H_b(p, b, v)m(p, b, v)$	$L_w(w, b, v) = -L_b(w, b, v)M(w, b, v)$
(7)	Expenditure function	$e(p, u) = \text{Min}_x\,[p \cdot x\colon u(x) \geqq u]$	$E(p, u, v) = \text{Min}_m\,[p \cdot m\colon U(m, v) \geqq u]$	$\bar{E}(w, u, v) = \text{Min}_M\,[w \cdot M\colon \bar{U}(M, v) \geqq u]$
(8)	→ Hicksian demand functions	$e_p(p, u) = x^c(p, u)$	$E_p(p, u, v) = m^c(p, u, v)$	$\bar{E}_w(w, u, v) = M^c(w, u, v)$
(9)	"Slutsky Identity"	$x^c(p, u) = x[p, e(p, u)]$	$m^c(p, u, v) = m[p, E(p, u, v), v]$	$M^c(w, u, v) = M[w, \bar{E}(w, u, v), v]$
(10)	→ Slutsky equation	$x^c_p = x_p + x_I x'$	$m^c_p = m_p + m_b m'$	$M^c_w = M_w + M_b M'$

The Supply Side

Consider a country in autarky. Assume perfectly competitive product and factor markets and CRS.

- Denote by V the endowment vector and $y(V)$ the vector of increasing and concave production functions.
- Denote by $g(p, V)$ the GDP function:

$$g(y(p, V)) = p.y(p, V) = \max\{p.y : \text{y is feasible given } v\}$$

- Graphically $g(\cdot)$ is the 'level' of a tangent to the PPF.
- Profit maximization implies that the MRT equals the relative price.
- So GDP is maximal and $g(\cdot)$ describes GDP in autarky.

Properties of $g(p, V)$

Assume $g(p, V)$ is twice differentiable in both arguments.

- g is homogenous of degree 1 in p: if all prices rise proportionally, optimal output remains constant and GDP rises proportionally.
- g is homogenous of degree 1 in v: if all factor endowments rise proportionally, optimal input use remains constant and costs rise proportionally.
- Euler theorem: $\frac{\partial g}{\partial p_i} = y_i$ and $\frac{\partial g}{\partial V_v} = \omega_v$.
- g is convex in p: if p increases, either g increases linearly or output is reallocated and g increases by more.
- g is concave in V: marginal productivity of factors is non-increasing.
- Hessian of $g(\cdot)$:

$$\begin{vmatrix} g_{pp} & g_{pV} \\ g_{Vp} & g_{VV} \end{vmatrix} = \begin{vmatrix} \left|\frac{\partial y}{\partial p}\right| & \left|\frac{\partial y}{\partial v}\right| \\ \left|\frac{\partial \omega}{\partial p}\right| & \left|\frac{\partial \omega}{\partial V}\right| \end{vmatrix}$$

- g_{pV}: Rybczynski derivatives, endowment-output responses
  g_{Vp}: Stolper-Samuelson derivatives, price-to-factor-price responses.

The Demand Side

Assume convex and homothetic preferences. Let $e(p, u)$ be the expenditure function and $h^c(p, u)$ the compensated demand function. Define $E(p, V, u)$ as the Trade Expenditure Function:

$$E(p, V, u) = e(p, u) - g(p, V)$$

Properties of $E(\cdot)$:

- $E_p(p, V, u) = e_p(p, u) - g_p(p, V) = h(p, u) - y(p, V) = m(p, u, V)$, the *compensated* excess demand function.
- $E(\cdot)$ solves the following program:

$$\min_m \{p.m(p, u, V) : \tilde{U}(m, V) \geq u\}$$

  $\tilde{U}(\cdot)$ is called the Meade utility function.
- $E(\cdot)$ is increasing in all its arguments.
- $E(\cdot)$ is concave in p, excess demand functions cannot slope upwards.

Application to the 2-sector Ricardian model

- The GDP function is written as

$$g(p, L) = \max_i \{\sum_i p_i \frac{L_i}{a_i}\}$$

- If $\frac{p_1}{a_1} > \frac{p_2}{a_2} \Leftrightarrow \frac{p_1}{p_2} > \frac{p_1^a}{p_2^a}$ then Home specializes in 1, otherwise in 2.
- This implies:

$$g(p, L) = \max_i \{\frac{p_i L}{a_i}\}; \qquad g^*(p, L^*) = \max_i \{\frac{p_i L^*}{a_i^*}\}$$

- The GDP function is convex in each price, with a kink due to the discontinuity.

The Hotelling and Shepard lemmas imply:

$$\forall i, \{g_{p_i}(p, L), g^*_{p_i}(p, L^*)\} = \{y_i, y_i^*\} = \begin{array}{ll} \{\frac{L}{a_i}, 0\} & \text{if } a_i < a_i^* \\ \{0, \frac{L^*}{a_i}\} & \text{if } a_i > a_i^* \end{array}$$

$$g_L(p, L) = w = \frac{p_i}{a_i}, \forall i$$

$$g_{L^*}(p, L^*) = w^* = \frac{p_i}{a_i^*}, \forall i$$

Goods and factor market clearing imply:

$$\{y_i, y_i^*\} = \begin{array}{ll} \{x_i(p) + x_i^*(p), 0\} & \text{if } a_i < a_i^* \\ \{0, x_i(p) + x_i^*(p)\} & \text{if } a_i > a_i^* \end{array}$$

$$a'.y = L$$

$$a^{*'}.y^* = L^*$$

Application to the HOS model

$$g(p, K, L) = p_1 f_1(K_1, L_1) + p_2 f_2(K_2, L_2)$$
$$g^*(p, K^*, L^*) = p_1 f_1(K_1^*, L_1^*) + p_2 f_2(K_2^*, L_2^*)$$

The Hotelling and Shepard lemmas imply:

$$\forall i, \{y_i, y_i^*\} = \{f_i(K_i, L_i), f_i(K_i^*, L_i^*)\}$$
$$\forall i, \{w, r\} = \{p_i a_{Li}(K, L), p_i a_{Ki}(K, L)\}$$
$$\forall i, \{w, r\} = \{p_i a_{Li}^*(K^*, L^*), p_i a_{Ki}^*(K^*, L^*)\}$$

Goods and factor market clearing imply:

$$\forall i, y_i(p) + y_i^*(p) = x_i(p) + x_i^*(p)$$
$$\begin{vmatrix} a_{L1} & a_{L2} \\ a_{K1} & a_{K2} \end{vmatrix} \cdot \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} = \begin{vmatrix} L \\ K \end{vmatrix}$$
$$\begin{vmatrix} a_{L1} & a_{L2} \\ a_{K1} & a_{K2} \end{vmatrix} \cdot \begin{vmatrix} y_1^* \\ y_2^* \end{vmatrix} = \begin{vmatrix} L^* \\ K^* \end{vmatrix}$$

Gains from Trade

- Autarky and free trade equilibria imply

$$m^a(p^a, u^a) = 0 = m^{ft}(p^{ft}, u^{ft})$$

- Consider $E(p^{ft}, u^a)$. A property of E is that:

$$E(p^{ft}, u^a) \leq p^{ft}.\bar{m}, \forall \bar{m} : \tilde{U}(\bar{m}) \geq u^a$$

  and since $\tilde{U}(\bar{m}) = u^a$ it must be that:

$$E(p^{ft}, u^a) \leq p^{ft}.m^a$$

- By expenditure minimization and GDP maximization in autarky

$$E(p^{ft}, u^a) \leq E(p^a, u^a)$$

- From equilibrium conditions $E(p^a, u^a) = E(p^{ft}, u^{ft}) = 0$ we have

$$E(p^{ft}, u^a) \leq E(p^{ft}, u^{ft}) \Rightarrow u^a \leq u^{ft}$$

  since E is an increasing function of u.
- Under free trade one can reach autarky utility with money to spare.

Conclusions on the 2x2x2 HOS Model

- General equilibrium trade model with sharp predictions on trade patterns.
- When endowments are in the FPE set, free trade in goods replicates an integrated economy.
- Can the model be generalized to more goods and factors?
  - equal number of goods and factors : relatively straightforward
  - more goods than factors : the factor content of trade can be predicted despite some indeterminacies
  - more factors than goods: too few ZP equations for the number of factor prices, but the model can be solved in special cases (Ricardo-Viner specific factors model).

Appendix: Factor Content of Consumption

- Under homothetic and identical preferences demand takes the form $x^c = \alpha(p)Y^c, c = H, F$, and the the factor content of each good's consumption takes the form $A'\alpha(p)Y^c$ in each country c.
- Ex.: if $U(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ then capital content of consumption in c equals $(\frac{a_{1K}\alpha}{p_1} + \frac{a_{2K}(1-\alpha)}{p_2})Y^c$
- Comparing capital contents across countries, they are in the same proportion than labor contents and income, e.g.

$$\frac{(\frac{a_{K1}\alpha}{p_1} + \frac{a_{K2}(1-\alpha)}{p_2})Y^H}{(\frac{a_{K1}\alpha}{p_1} + \frac{a_{K2}(1-\alpha)}{p_2})Y^F} = \frac{(\frac{a_{L1}\alpha}{p_1} + \frac{a_{L2}(1-\alpha)}{p_2})Y^H}{(\frac{a_{L1}\alpha}{p_1} + \frac{a_{L2}(1-\alpha)}{p_2})Y^F} = \frac{Y^H}{Y^F}$$

- From (FE) they must be in the same proportion as world endowments, hence on the diagonal of the FPE set.

Appendix: Gains From Trade

- Traditional gains from trade argument:

$e(p^{ft}, u^a) \leq (p^{ft})' c^a$	expenditure min.
$(p^{ft})' c^a = (p^{ft})' y^a$	GM clearing in autarky
$(p^{ft})' y^a \leq (p^{ft})' y^{ft}$	GDP max.
$(p^{ft})' y^{ft} = (p^{ft})' c^{ft} = e(p^{ft}, u^{ft})$	trade balance, expenditure min.
$\Rightarrow e(p^{ft}, u^a) \leq e(p^{ft}, u^{ft}) \Rightarrow u^a \leq u^{ft}$	e increasing in u

- Trade Expenditure formulation $E(p, u, V) \equiv e(p, u) - g(p, V)$

$E(p^{ft}, u^a) = (p^{ft})' m(p^{ft}, u^a, V)$	Euler theorem, $E_p = m$
$(p^{ft})' m(p^{ft}, u^a, V) \leq (p^{ft})' m(p^a, u^a, V)$	property of E, $\tilde{U}(m^a) = u^a$
$(p^{ft})' m(p^a, u^a, V) \leq (p^a)' m(p^a, u^a, V)$	expenditure min., GDP max.
$(p^a)' m(p^a, u^a, V) = 0 = (p^{ft})' m(p^{ft}, u^{ft}, V)$	GM clearing, trade balance
$\Rightarrow E(p^{ft}, u^a) \leq E(p^{ft}, u^{ft}) \Rightarrow u^a \leq u^{ft}$	E increasing in u

- Inequalities are weak because IC or PPF may not be strictly convex.